Using Intrelab to Quickly Prototype Interface Algorithms

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Trilinos User Group, CSRI 90
October 29, 2014

SAND2014-19921 C
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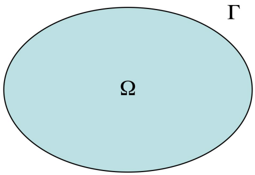
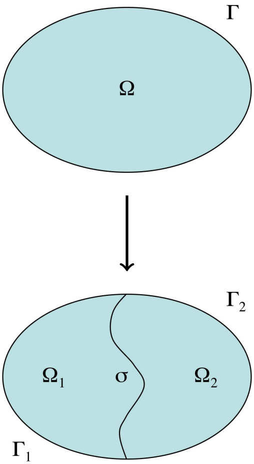
Domain Layout



Domain Layout



Linear Elasticity on Ω

- Governing Equations

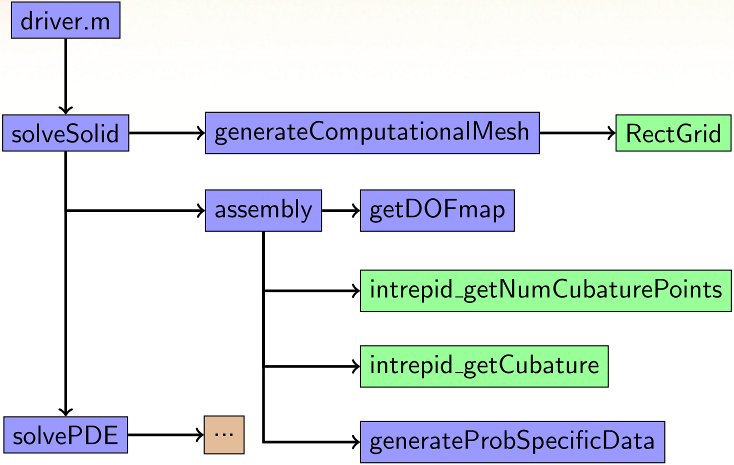
$$\rho \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \nabla \cdot \sigma(\boldsymbol{\eta}) = f \quad in\, \Omega$$

- We get a variational form by multiplying by a test function, integrating over Ω, and discretizing in time by the central difference scheme (explicit) and in space with $\mathbb{P}^1$ finite elements,

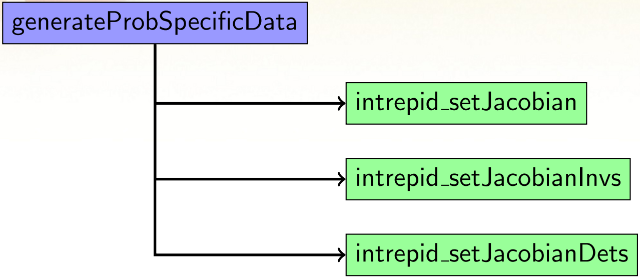
$$\int_\Omega \rho \frac{(\boldsymbol{\eta}_h^{n+1} - 2\boldsymbol{\eta}_h^{n} + \boldsymbol{\eta}_h^{n-1})}{\Delta t^2} \cdot \boldsymbol{\xi}_h + \sigma(\boldsymbol{\eta}_h^n) : \nabla \boldsymbol{\xi}_h \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\xi}_h \, dx \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{V}^h \subset \boldsymbol{H}^1(\Omega)$$

- To create a light framework for implementing interface algorithms we at least will require
  - Basis functions (Values, Gradients, Divergence)
  - Jacobians and their determinants
  - Quadrature rules

Workflow



[mesh] = RectGrid(xmin, xmax, ymin, ymax, nxint, nyint, ‘Triangle’);
intrepid_getNumCubaturePoints(‘Triangle’, cubDegree);
intrepid_getCubature(cubPoints, cubWeights, ‘Triangle’, cubDegree);




```
% evaluate cell Jacobians
cellJacobians = zeros(spaceDim, spaceDim, numCubPoints, numCells);
intrepid_setJacobian(cellJacobians, cubPoints, ...
        cellNodes, 'Triangle');
% evaluate inverses of cell Jacobians
cellJacobianInvs = zeros(spaceDim, spaceDim, numCubPoints, numCells);
intrepid_setJacobianInv(cellJacobianInvs, cellJacobians);
% evaluate determinants of cell Jacobians
cellJacobianDets = zeros(numCubPoints, numCells);
intrepid_setJacobianDet(cellJacobianDets, cellJacobians);
```

generateProbSpecificData (Reference Cell Evaluations)

```
% evaluate basis (value, gradient)
val_at_cub_points = zeros(numCubPoints, numFields);
grad_at_cub_points = zeros(spaceDim, numCubPoints, numFields);
intrepid_getBasisValues(val_at_cub_points, cubPoints, ...
        'OPERATOR_VALUE', 'Triangle', 1);
intrepid_getBasisValues(grad_at_cub_points, cubPoints, ...
        'OPERATOR_GRAD', 'Triangle', 1);

% compute cell measures
weighted_measure = zeros(numCubPoints, numCells);
intrepid_computeCellMeasure(weighted_measure, ...
        cellJacobianDets, cubWeights);

% compute cell volumes
cell_volumes = sum(weighted_measure,1);
```

generateProbSpecificData (Transform Values)

$$\int_{Phys(\mathcal{T})} \phi_j \cdot \phi_i \; dx = \int_{Ref(\mathcal{T})} \hat{\phi}_j \cdot \hat{\phi}_i \; det(J) d\hat{x}$$

```
% transform values
transformed_val_at_cub_points = ...
        zeros(numCubPoints, numFields, numCells);
intrepid_HGRADtransformVALUE(transformed_val_at_cub_points, ...
        val_at_cub_points);

% combine transformed values with measures
weighted_transformed_val_at_cub_points = ...
        zeros(numCubPoints, numFields, numCells);
intrepid_multiplyMeasure(weighted_transformed_val_at_cub_points, ...
        weighted_measure, transformed_val_at_cub_points);
```

generateProbSpecificData (Transform Gradient Values)

$$\int_{Phys(\mathcal{T})} \nabla\phi_j : \nabla\phi_i \, dx = \int_{Ref(\mathcal{T})} J^{-T}\hat{\nabla}\hat{\phi}_j : J^{-T}\hat{\nabla}\hat{\phi}_i \, det(J) d\hat{x}$$

```
% transform gradients
transformed_grad_at_cub_points = zeros(spaceDim, numCubPoints, ...
        numFields, numCells);
intrepid_HGRADtransformGRAD(transformed_grad_at_cub_points, ...
        cellJacobianInvs, grad_at_cub_points);

% combine transformed gradients with measures
weighted_transformed_grad_at_cub_points = zeros(spaceDim, ...
        numCubPoints, numFields, numCells);
intrepid_multiplyMeasure(weighted_transformed_grad_at_cub_points, ...
        weighted_measure, transformed_grad_at_cub_points);
```

generateProbSpecificData (Mass Matrix)

$$\int_{Phys(\mathcal{T})} \phi_j \cdot \phi_i \; dx = \int_{Ref(\mathcal{T})} \hat{\phi}_j \cdot \hat{\phi}_i \; det(J) d\hat{x}$$

```
% integrate mass matrix
cell_mass_matrices = zeros(numFields, numFields, numCells);
intrepid_integrate(cell_mass_matrices, ...
        transformed_val_at_cub_points, ...
        weighted_transformed_val_at_cub_points, 'COMP_BLAS');

% build global mass matrix
cell_mass_matrices_reshape = ...
        reshape(cell_mass_matrices, 1, numel(cell_mass_matrices));
mass_mat = sparse(iIdxVertices,jIdxVertices,
        repmat(cell_mass_matrices_reshape',2,1));
```

generateProbSpecificData (Stiffness Matrix)

$$\int_{Phys(\mathcal{T})} \nabla\phi_j : \nabla\phi_i \; dx = \int_{Ref(\mathcal{T})} J^{-T}\hat{\nabla}\hat{\phi}_j : J^{-T}\hat{\nabla}\hat{\phi}_i \; det(J) d\hat{x}$$

```
% integrate stiffness matrix
cell_deformation_matrices = zeros(numFields, numFields, numCells);
intrepid_integrate(cell_deformation_matrices, ...
        transformed_grad_at_cub_points, ...
        weighted_transformed_grad_at_cub_points, 'COMP_BLAS');

% build global stiffness matrix
cell_deformation_matrices_reshape = ...
        reshape(cell_deformation_matrices, 1, ...
        numel(cell_deformation_matrices));
deformation_mat = sparse(iIdxVertices,jIdxVertices, ...
        repmat(cell_deformation_matrices_reshape',2,1));
```

Rewriting the weak form of the problem

We now have our Mass Matrix ($\mathbf{M}$) and our Stiffness Matrix ($\mathbf{K}$) and we can rewrite

$$\int_\Omega \rho \frac{(\boldsymbol{\eta}_h^{n+1} - 2\boldsymbol{\eta}_h^{n} + \boldsymbol{\eta}_h^{n-1})}{\Delta t^2} \cdot \boldsymbol{\xi}_h + \sigma(\boldsymbol{\eta}_h^n) : \nabla \boldsymbol{\xi}_h \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\xi}_h \, dx \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{V}^h \subset \boldsymbol{H}^1(\Omega)$$

as

$$\frac{\rho}{\Delta t^2} \mathbf{M} \vec{\boldsymbol{\eta}}^{n+1} = \frac{\rho}{\Delta t^2} \mathbf{M} (2\vec{\boldsymbol{\eta}}^{n} - \vec{\boldsymbol{\eta}}^{n-1}) - K \vec{\boldsymbol{\eta}}^{n} + \mathbf{M} \vec{\boldsymbol{f}}^{n}$$

We use a diagonal mass matrix (nodal quadrature), and therefore we can take care of boundary conditions by using this relation for all non-Dirichlet boundary nodes, and use the prescribed value at Dirichlet boundary nodes.

$$\boxed{\text{solvePDE}} \longrightarrow \boxed{\text{solveSolid}} \longrightarrow \boxed{\text{solvePDE}}$$

- Calculate initial conditions
- $\boxed{\text{exchange}}$ interface information
- Loop over time steps using relation

$$\frac{\rho}{\Delta t^2}\mathbf{M}\vec{\eta}^{n+1} = \frac{\rho}{\Delta t^2}\mathbf{M}(2\vec{\eta}^{n} - \vec{\eta}^{n-1}) - K\vec{\eta}^{n} + \mathbf{M}\vec{\boldsymbol{f}}^{n}$$

- End with $\boxed{\text{computeError}}$ if there is a known solution

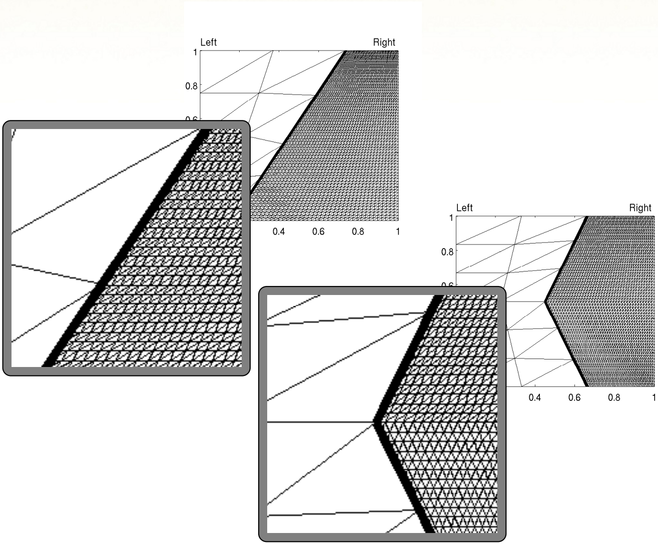
computeError

```
% set up more accurate numerical integration
cubDegree = 6;
```

Then, get

- cubature points and weights
- Jacobians, Jacobian inverses, and determinants of Jacobians
- values and gradients on reference cells
- transformed values and gradients
- combined transformed values and gradients with measures
- ...

Computional Meshes



Convergence Rates

(Using P1 Elements)

LEFT mesh	L2 error	L2 ratio	L2 rate	H1 error	H1 ratio	H1 rate
0016x0010	4.45e-02	-	-	1.62e+00	-	-
0024x0015	2.51e-02	0.56	1.42	1.10e+00	0.68	0.96
0032x0020	1.64e-02	0.65	1.48	8.17e-01	0.74	1.04
0048x0030	9.18e-03	0.56	1.43	5.54e-01	0.68	0.96
0064x0040	5.56e-03	0.61	1.74	4.05e-01	0.73	1.09
0096x0060	2.99e-03	0.54	1.53	2.70e-01	0.67	0.99
0144x0090	1.44e-03	0.48	1.80	1.76e-01	0.65	1.06
0208x0130	5.94e-04	0.41	2.40	1.17e-01	0.66	1.11

RIGHT mesh	L2 error	L2 ratio	L2 rate	H1 error	H1 ratio	H1 rate
0014x0016	8.97e-02	-	-	3.87e+00	-	-
0021x0024	4.28e-02	0.48	1.83	2.46e+00	0.64	1.11
0028x0032	2.68e-02	0.63	1.63	1.87e+00	0.76	0.95
0042x0048	1.27e-02	0.47	1.85	1.23e+00	0.66	1.04
0056x0064	7.73e-03	0.61	1.72	9.31e-01	0.76	0.96
0084x0096	3.83e-03	0.50	1.73	6.18e-01	0.66	1.01
0126x0144	1.71e-03	0.45	2.00	4.09e-01	0.66	1.02
0182x0208	6.99e-04	0.41	2.42	2.79e-01	0.68	1.03

FULL mesh	L2 error	L2 ratio	L2 rate	H1 error	H1 ratio	H1 rate
0030x0016	7.51e-02	-	-	3.62e+00	-	-
0045x0024	3.40e-02	0.45	1.95	2.35e+00	0.65	1.07
0060x0032	1.93e-02	0.57	1.98	1.75e+00	0.74	1.03
0090x0048	8.61e-03	0.45	1.99	1.16e+00	0.66	1.02
0120x0064	4.86e-03	0.56	1.99	8.65e-01	0.75	1.01
0180x0096	2.17e-03	0.45	1.99	5.76e-01	0.67	1.00
0270x0144	9.72e-04	0.45	1.98	3.84e-01	0.67	1.00
0390x0208	4.73e-04	0.49	1.96	2.66e-01	0.69	1.00