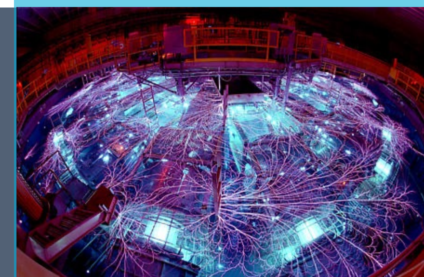
On Scalable Multiphysics Block Preconditioning of an Implicit VMS Resistive MHD Formulation with Application to MCF*



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Fusion Energy Sciences

Sandia National Laboratories

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Three useful resources on fusion energy for students:

Contemporary Physics Education Project https://www.cpepphysics.org/

Princeton Plasma Physics Laboratory
2021 Introduction to Fusion Energy and Plasma Physics Course
https://suli.pppl.gov/2021/course/

Progress toward fusion energy breakeven and gain as measured against the Lawson criterion
Wurzel, Hsu, Physics of Plasmas 29, 062103 (2022)

Outline

- **MCF Motivation**
- **VMS FE MHD Formulation**
- **A few ITER relevant results**
- **Multiphysics Block Preconditioner**
- **Performance of preconditioner**
- **Conclusions**

A Very Few Comments on MCF Energy

Why fusion power?

Source: Cami Collins, Oak Ridge National Laboratory

To produce 1000 megawatts electricity for 1 day (enough for a major city)

Coal Plant				D-T Fusion Plant
	*18 million Lb coal* **80** **railroad cars**	**Fuel Consumed**	*1.0 Lb D_2* *1.5 Lb T_2* **3** **water bottles**	
	*61 million Lb greenhouse gases* **28,000** **33 foot spheres**	**Waste Produced**	*2.0 Lb helium* **400** **balloons**	

Why fusion power? Energy Release!

To produce 1000 megawatts electricity for 1 day (enough for a major city)

Energy Density Comparison

DT Fusion: **339** GJ/g DT

Fission: **82** GJ/g U-235

Methane: **20** kJ/g CH_4

Fusion is a million times larger!

D-T Fusion Plant

1.0 Lb D_2
1.5 Lb T_2

3
water bottles

2.0 Lb
helium

400
balloons

Source: Cami Collins, Oak Ridge National Laboratory

How close are we to fusion power?

Fusion Triple product $(n_{i0}\ T_{i0}\ \tau_E^*)$: MCF Energy released in fusion products must exceed total energy applied as heat.

Progress toward fusion energy breakeven and gain as measured against the Lawson criterion

Cite as: Phys. Plasmas 29, 062103 (2022); https://doi.org/10.1063/5.0083990
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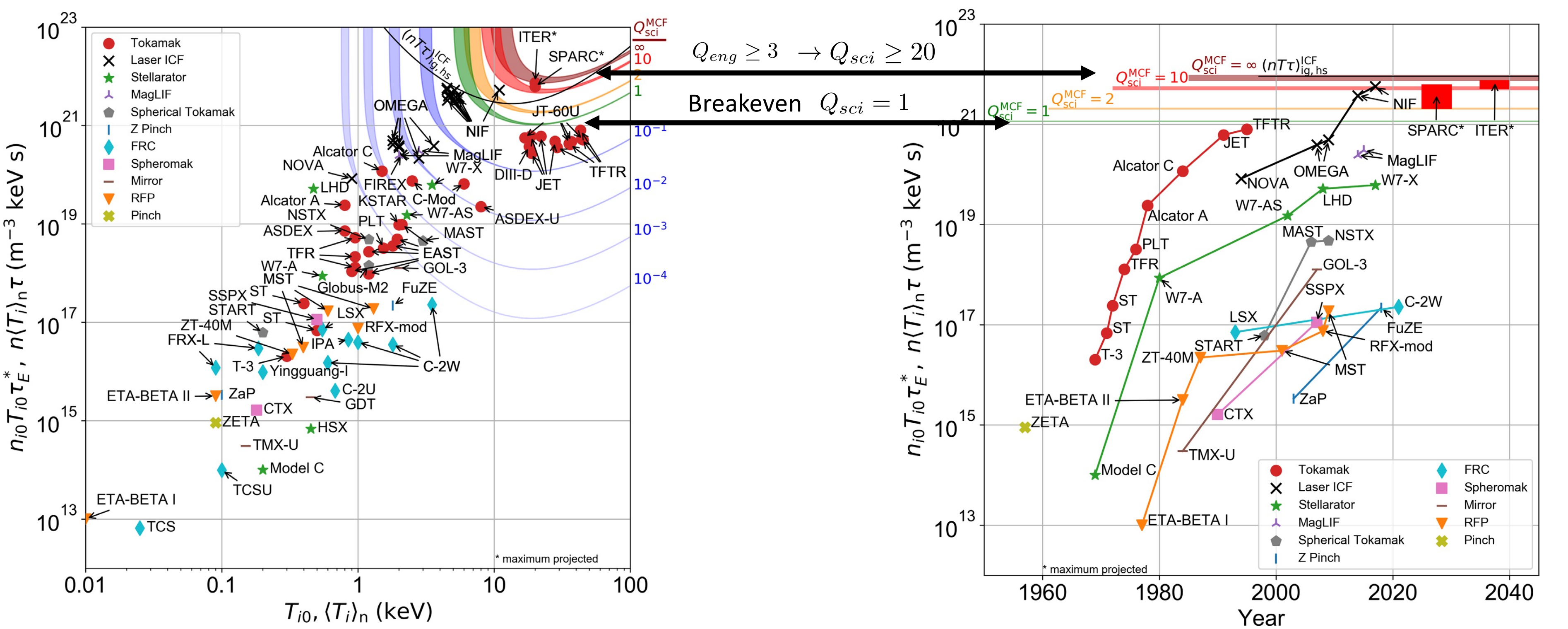
Samuel E. Wurzel and Scott C. Hsu

$$Q_{\text{sci}} = \frac{P_{\text{out}} - P_{\text{ext}}}{P_{\text{ext}}} = \frac{P_F}{P_{\text{ext}}},$$

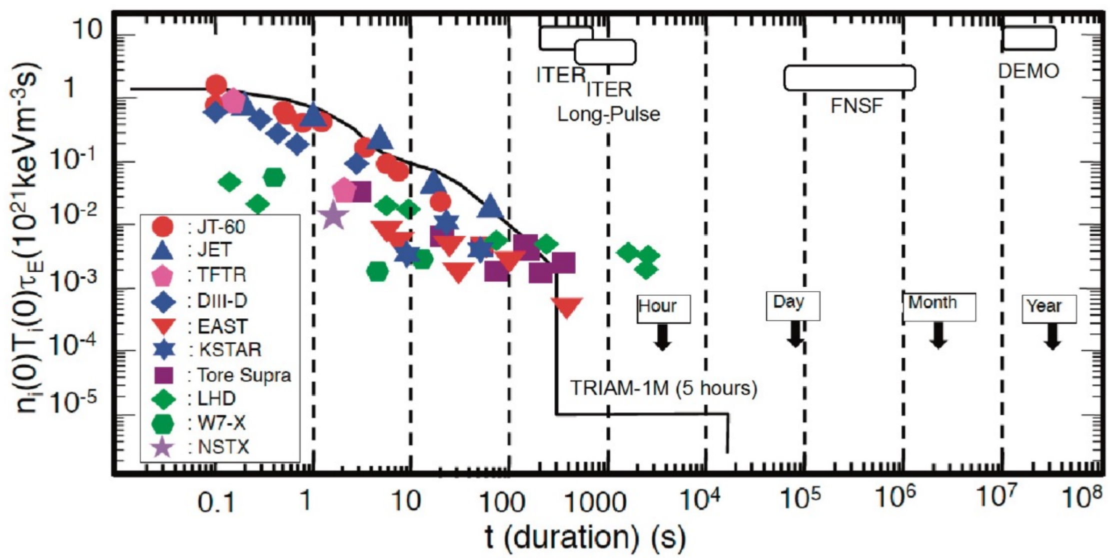


Credit right image: Cami Collins, Oak Ridge National Laboratory

Source: Cami Collins, Oak Ridge National Laboratory

But Making Electricity Is More Than Just Triple Product

Fusion Triple Product vs. Plasma Duration



Significant progress is needed to demonstrate high gain AND long-duration (or high rep rate?) to be relevant for cost-effective, uninterrupted fusion power production

For Sufficiently Long MCF Plasma Confinement Times Understanding and Controlling Instabilities/Disruptions in Plasma Confinement is Critical.

Goal for Fusion Device:

- Achieve temperatures of > 100M deg K (> 6x Sun temp.) ,
- Burning plasma / energy confinement times of O(1) – O(10) sec. (ITER).

Strong external magnetic fields are used for:

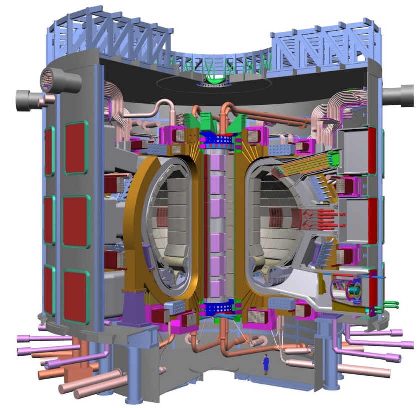
- Resistive heating of the plasma (along with RF-EM waves, ..)
- Confinement of the hot plasma to keep it from striking the wall

Plasma disruptions can

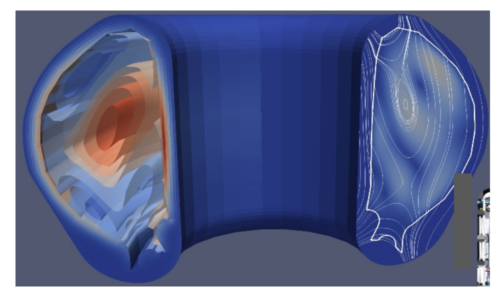
- cause a loss of vertical positioning control,
- a breakdown of magnetic confinement with huge plasma thermal energy loss to the walls, and
- a discharge of very large electrical currents to surface,

that can damage the device.

ITER can sustain only a limited number of significant disruptions, O(1 – 5) without bringing down the device.



ITER Tokamak [under construction, Cadarache, Fr.]

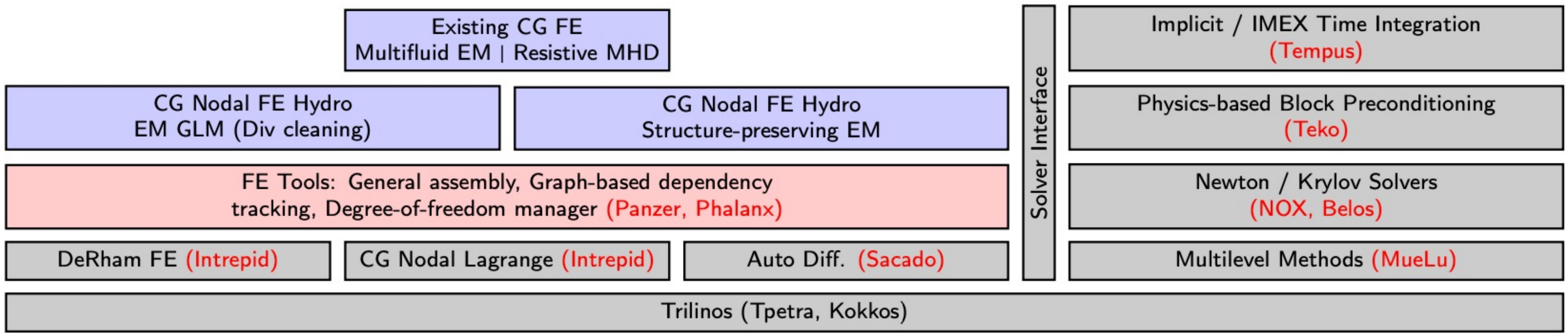


Vertical Displacement Event (VDE) in ITER Tokamak Plasma and Wall Geometry (Drekar sim.)

Context

Drekar: Resistive MHD / Multifluid with Coupled Multiphysics

- Arbitrarily many equations describing physics (continuity, momentum, energy, electromagnetics).
- ERK, DIRK, IMEX time integration (Tempus).
- 2D & 3D unstructured finite element (Intrepid):
  - Stabilized Q1/P1 elements (high-order possible).
  - Physics compatible discretizations (node, edge, face).
  - High-resolution positivity-preserving methods.
- Advanced software capabilities:
  - MPI+X (Kokkos).
  - Linear/non-linear solvers (NOX, Belos) with robust, scalable preconditioning (Teko, MueLu).
  - Jacobians computed through automatic differentiation (Sacado).
  - Asynchronous dependency manages multiphysics complexity (Phalanx).



Trilinos Assembly/Evaluation Engines

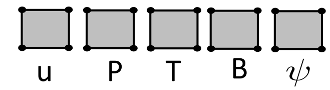
Panzer: Multiphysics finite element assembly engine.

- Implement models using *equation set* classes to describe physics in residual form (eg., weak form residual).
- Manages arbitrary assignments of physics models (*equation sets*) to mesh regions (*element blocks*) with various discretizations.
- Handles indexing of solution fields into global solution vectors, Jacobian matrices, etc.

Phalanx: DAG-based expression evaluation.

- Each node (*evaluator*) maps input fields to output fields (Ideal gas EoS: $(\rho, \rho\mathbf{u}, \mathcal{E}) \mapsto (p, T)$).
- Written using *evaluate* strategy $(\text{output} = f(\text{input}))$ or *contribute* strategy $(\text{output} += f(\text{input}))$.
- Simple closure relations (eg., equation of state) leverage *evaluate* strategy for flexibility: just replace with a different evaluation.
- *Contribute* strategy allows for flexibility in model construction: *evaluate* a base model, then *contribute* specialized components for specific models.
- Template evaluators on scalar type to support generation of Jacobian matrices through automatic differentiation (AD).

Basic Resistive Low Mach Number Magnetohydrodynamics (MHD) is “useful” for Studying Some Aspects of Macroscopic Instabilities and Disruptions in MCF



Conservation of mass

$$\frac{\partial \rho}{\partial t} + \boxed{\nabla \cdot (\rho \mathbf{u})} = 0,$$

Sound wave off diagonal coupling

Lorentz force

Conservation of momentum

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\boxed{(\rho \mathbf{u} \otimes \mathbf{u})} + pI + \frac{2}{3}\frac{1}{Re}(\nabla \cdot \mathbf{u})I - \boxed{\frac{1}{Re}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)} \right] - \boxed{\mathbf{j} \times \mathbf{B}} = \mathbf{0},$$

Internal energy balance eq.

$$\frac{\partial T}{\partial t} + \boxed{\mathbf{u} \cdot \nabla T} + \frac{2}{3}T(\nabla \cdot \mathbf{u}) + \boxed{\nabla \cdot \mathbf{q}} = 0,$$

Alfven wave strong off diagonal coupling

Magnetic induction

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left[\boxed{\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}} - \boxed{\frac{1}{S}(\nabla \mathbf{B} - (\nabla \mathbf{B})^T)} + \psi I \right] = \mathbf{0},$$

Solenoidal constraint

$$\nabla \cdot \mathbf{B} = 0,$$

Elliptic divergence cleaning

Source term

Convection

Diffusion

Cross-coupled terms

Whole Device Modeling Requires Heterogeneous Multiphysics

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} - (\mathbf{T} + \mathbf{T}_M)] + 2\rho\Omega \times \mathbf{v} - \rho \mathbf{g} = \mathbf{0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\mathbf{T} = -\left(P + \frac{2}{3}\mu(\nabla \bullet \mathbf{u})\right)\mathbf{I} + \mu[\nabla \mathbf{u} + \nabla \mathbf{u}^T]$$

$$\mathbf{T_M} = \frac{1}{\mu_0}\mathbf{B} \otimes \mathbf{B} - \frac{1}{2\mu_0}\|\mathbf{B}\|^2\mathbf{I}$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot [\rho \mathbf{v} e + \mathbf{q}] - \mathbf{T} : \nabla \mathbf{v} - \eta \| \frac{1}{\mu_0} \nabla \times \mathbf{B} \|^2 = 0$$

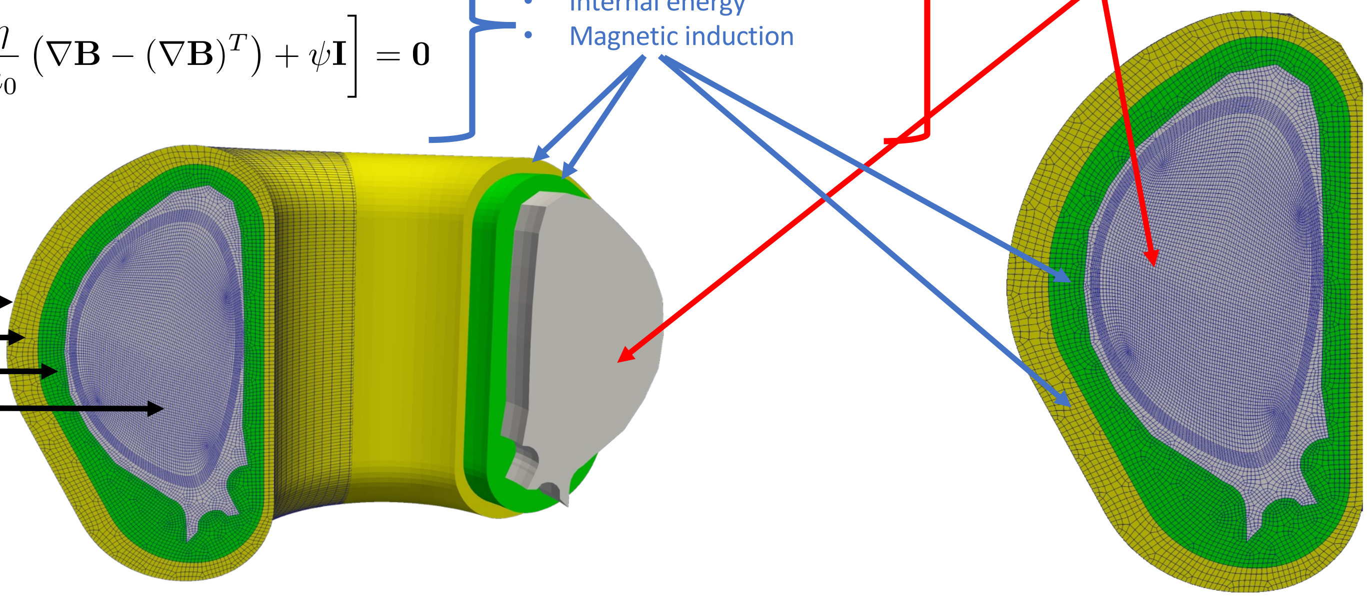
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left[\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B} - \frac{\eta}{\mu_0}\left(\nabla \mathbf{B} - (\nabla \mathbf{B})^T\right) + \psi \mathbf{I}\right] = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = 0$$

Plasma region: visco-resistive MHD
- Momentum
- Mass
- Internal energy
- Magnetic induction

Solid walls and vacuum regions
- Internal energy
- Magnetic induction

Heterogeneous Physics Regions
- Outer perfect conducting wall
- Vacuum region
- Solid 1st wall
- Plasma region



Discretization

Finite element discretization (Galerkin terms)

Find $\mathbf{U} \doteq [\rho, \mathbf{m}, T, \mathbf{B}, \psi]^T \in \mathcal{U}$ such that $\rho = \overline{\rho}$ on Γ_D^{ρ}, $\mathbf{m} = \overline{\mathbf{m}}$ on $\Gamma_D^{\mathbf{m}}$, $T = \overline{T}$ on Γ_D^{T}, $\mathbf{B} = \overline{\mathbf{B}}$ on $\Gamma_D^{\mathbf{B}}$, $\psi = \overline{\psi}$ on Γ_D^{ψ}, and

$$\mathcal{A}(\mathbf{W}, \mathbf{U}) = \mathcal{F}(\mathbf{W}) \quad \forall\ \mathbf{W} \doteq [q, \mathbf{w}, \theta, \mathbf{C}, s]^T \in \mathcal{V},$$

where

$$\begin{aligned}
\mathcal{A}(\mathbf{W}, \mathbf{U}) \doteq{} & (q, \partial_t \rho) - (\nabla q, \rho \mathbf{u}) \\
& + (\mathbf{w}, \partial_t \rho \mathbf{u}) - \langle \nabla \mathbf{w}, \rho \mathbf{u} \otimes \mathbf{u} \rangle - (\nabla \cdot \mathbf{w}, p + \tfrac{2}{3Re}(\nabla \cdot \mathbf{u})) + \langle \nabla \mathbf{w}, \tfrac{1}{Re}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \rangle - (\mathbf{w}, \mathbf{j} \times \mathbf{B}) \\
& + (\theta, \partial_t T) + (\theta, \mathbf{u} \cdot \nabla T) + \frac{2}{3}(\theta, T(\nabla \cdot \mathbf{u})) + (\nabla \theta, \mathbf{q}) \\
& + (\mathbf{C}, \partial_t \mathbf{B}) - \langle \nabla \mathbf{C}, \mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u} \rangle + \left\langle \nabla \mathbf{C}, \tfrac{1}{S} \left(\nabla \mathbf{B} - (\nabla \mathbf{B})^T \right) \right\rangle - (\nabla \cdot \mathbf{C}, \psi) \\
& + (s, \nabla \cdot \mathbf{B}),
\end{aligned}$$

$$(a, b) = \int_{\Omega} ab \ d\Omega$$

$$(\mathbf{a}, \mathbf{b}) = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \ d\Omega$$

$$\langle \mathbf{A}, \mathbf{B} \rangle = \int_{\Omega} \mathbf{A} : \mathbf{B} \ d\Omega$$

Deficiencies of Galerkin Weak Form:

- Equal-order interpolation have stability problems for saddle point prbs. (LBB condition, see .e.g. Gunzburger 1989)
  - Induction – div B = 0; Lagrange multiplier coupling $(\mathbf{B}, \psi)$
  - Strong guide field (large **B**) produces an incompressible flow limit type response plane $\perp \mathbf{B}$ and therefore, a saddle point like structure (e.g. Stokes-like behavior for $(\rho\mathbf{u}, \rho)$)

- Strong convective transport and large unresolved gradients can produce unphysical spatial oscillations (internal / boundary layers).

- For unresolved high-wavenumber signals aliasing of energy into lower-wavenumber resolved components

Brief Outline Following Variational Multiscale (VMS) Approach
VMS: T.J.R Hughes et. al.; & VMS MHD: Codina et. al., JS et. al.

Upwinding and saddle point stabilization

Split the solution and test spaces in resolved and unresolved scales, i.e., $\mathcal{U} = \mathcal{U}_h + \mathcal{U}'$ and $\mathcal{V} = \mathcal{V}_h + \mathcal{V}'$ thus we have

$$\mathcal{A}(\mathbf{W}_h, \mathbf{U}_h + \mathbf{U}') = \mathcal{F}(\mathbf{W}_h) \quad \forall\, \mathbf{W}_h \in \mathcal{V}_h$$

$$\cancel{\mathcal{A}(\mathbf{W}', \mathbf{U}_h + \mathbf{U}') = \mathcal{F}(\mathbf{W}') \quad \forall\, \mathbf{W}' \in \mathcal{V}'} \longrightarrow \mathbf{U}' \text{ not resolved, modeled by } \mathbf{U}' \approx -\boldsymbol{\tau}\mathbf{P}\mathcal{R}(\mathbf{U}^h)$$

I.e. sub-grid / unresolved scales driven by residual resolved scales of strong from PDEs, variationally consistent

VMS + additional optional DCO terms are included for enhanced stability

$$\left.\begin{aligned}
\mathcal{A}(\mathbf{W}_h, \mathbf{U}_h + \mathbf{U}') = \mathcal{A}(\mathbf{W}_h, \mathbf{U}_h) - \sum_{K \in \mathcal{T}_h} &\big((\nabla q_h, \rho_h \mathbf{u}' + \mathbf{u}_h \rho')_K \\
&+ \langle \nabla \mathbf{w}_h, \rho \mathbf{u}' \otimes \mathbf{u}_h + \rho' \mathbf{u}_h \otimes \mathbf{u}_h \rangle_K + (\nabla \cdot \mathbf{w}_h, p')_K \\
&+ \boxed{(\nabla \theta_h, \mathbf{u}_h T')_K} \\
&+ \langle \nabla \mathbf{C}_h, \mathbf{u}_h \otimes \mathbf{B}' - \mathbf{B}' \otimes \mathbf{u}_h \rangle_K + (\nabla \cdot \mathbf{C}_h, \psi')_K \\
&+ \boxed{(\nabla s_h, \mathbf{B}')_K}
\end{aligned}\right\}\text{VMS}$$

$$\left.+ (\nabla q_h, \nu_\rho^K \nabla \rho_h)_K \right\}\text{DCO LLF [density]}$$

$$\left.+ (\nabla \mathbf{w}_h, \tfrac{1}{2}\nu_{\mathbf{m}}^K (\nabla \mathbf{u}_h + (\nabla \mathbf{u}_h)^T))_K + \langle \nabla \mathbf{w}_h, \nu_\rho^K \nabla \rho_h \otimes \mathbf{u}_h \rangle_K \right\}\text{DCO R(s) [momentum]}$$

$$\left.+ \frac{C}{2}(u_A h\, \hat{b} \cdot \nabla \theta, \hat{b} \cdot \nabla T)_K \right\}\hat{b}\text{ DCO [temperature]}$$

First order cG finite elements for $\rho_h, \mathbf{m}_h, \mathbf{B}_h, \psi_h$ and **second order** finite elements T_h.

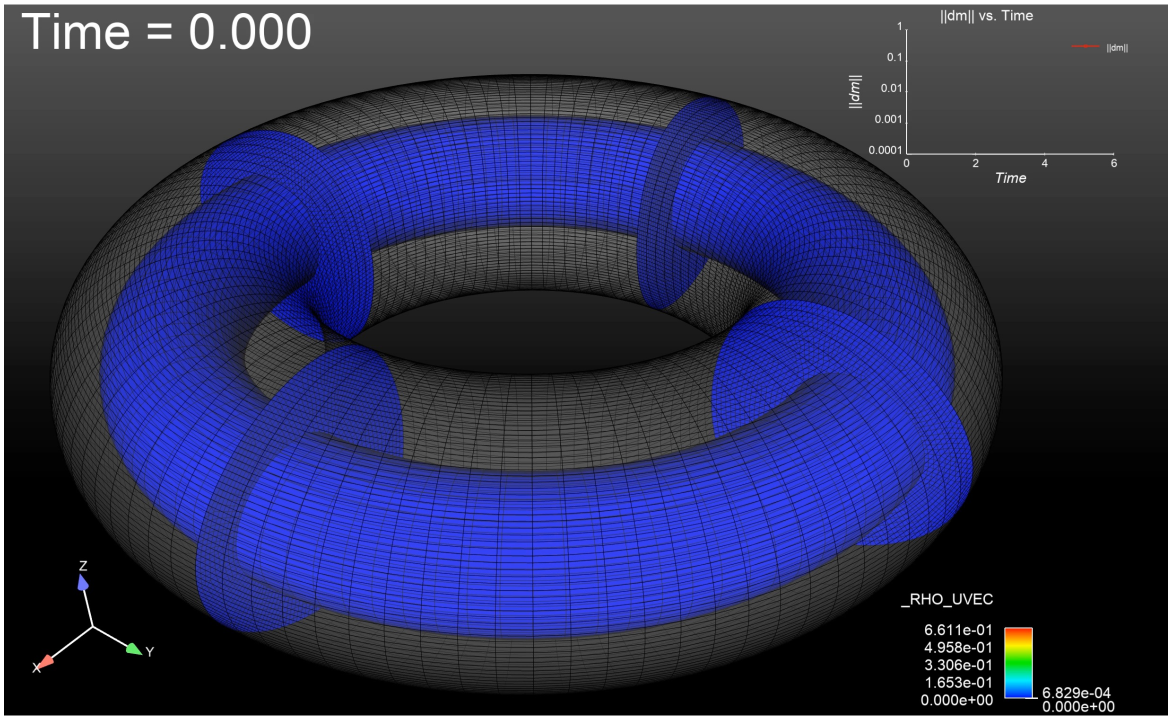
(momentum / density fluctuation as in X. Zeng, G. Scovazzi, A variational multiscale finite element method for monolithic ALE computations of shock hydrodynamics using nodal elements, J. Comput. Phys. 315 (2016) 577–608.)

Bonilla, S, Tang, Crockatt, Ohm, Phillips, Pawlowski, Conde, Beznosov, On a Fully-implicit VMS-stabilized FE Formulation for Low Mach Number Compressible Resistive MHD with Application to MCF, Comput. Methods Appl. Mech. Engrg. 2023

DCO: Hughes et. al, Tezduyar et. al.,
Entropy viscosity: Guermond et. al.

An example of the importance of implicit, implicit/explicit (IMEX) methods for over-stepping time-scales in magnetic confinement fusion relevant applications.

Resistive MHD: Soloveev Analytic Equilibrium
Nonlinear Disturbance Saturation (VMS Q1).



Kink and interchange instability.

MHD Wave speeds

$\|\mathbf{u}\|, \|\mathbf{u}\| \pm c_s, \|\mathbf{u}\| \pm c_a, \|\mathbf{u}\| \pm c_f, \pm c_h$ Here c_h is ∞ for elliptic divergence cleaning

Approx. Computational Time Scales:

- B Divergence Const. ($\nabla \cdot \mathbf{B} = 0$): $1/\infty = 0$
- Fast Magnetosonic Wave (c_f): 10^{-4} to 10^{-7}
- Alfven Wave (c_a): 10^{-4} to 10^{-7}
- Slow Magnetosonic Wave (c_s): 10^{-2} to 10^{-3}
- Sound Wave (c): 10^{-1} to 10^{-3}
- Convection ($c_{v\ max}$): $\sim 10^{-2}$
- Diffusion: 10^{-3} to 10^{-2}
- Macroscopic Dynamic Time-scale:
  unstable mode: O(1)

- Implicit time step $\Delta t = 10^{-2}$

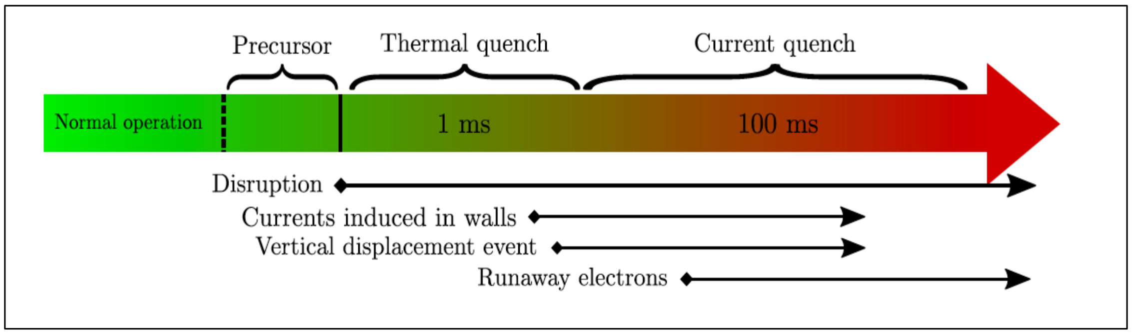
Fully-implicit (BDF2, SDIRK22)
Max CFL:

$CFL_{div} = \infty$
$CFL_{cf} \sim 10^5$
$CFL_{cA} \sim 10^5$
$CFL_{cs} \sim 10^1$
$CFL_{c} \sim 10^1$
$CFL_{cv} \sim 1$

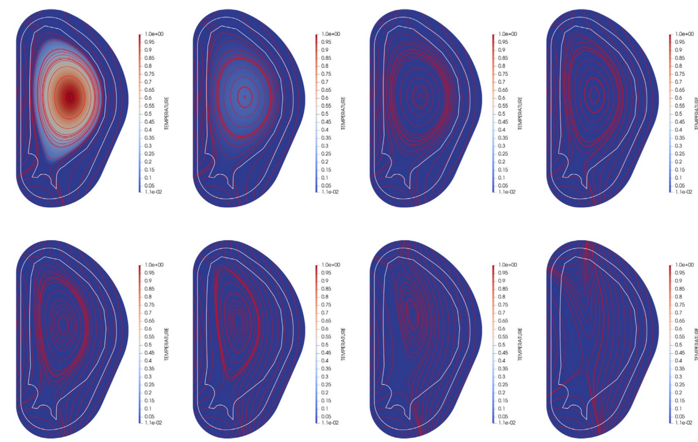
Preliminary Tokamak Relevant Results

Vertical displacement events (VDEs) are major disruption events occurring in tokamaks when vertical stability control is lost.

These events cause large currents to flow in the vessel and other adjacent metallic structures.



Drekar: Cold VDE in ITER



Cold VDE fast internal energy loss (i.e. Temperature drop)

1. **Initial equilibrium momentum force balance:** $\mathbf{u}_0 = 0$ **and** $\nabla P_0 = (\mathbf{j} \times \mathbf{B})_0$

$$\underbrace{\frac{\partial(\rho \mathbf{u})}{\partial t}}_{0} + \nabla \cdot \left[(\underbrace{\rho \mathbf{u} \otimes \mathbf{u}}_{0}) + pI + \frac{2}{3}\frac{1}{Re}(\underbrace{\nabla \cdot \mathbf{u}}_{0})I - \frac{1}{Re}(\underbrace{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}_{0}) \right] - \mathbf{j} \times \mathbf{B} = \mathbf{0}$$

2. **Temperature drops quickly, pressure changes** $\longrightarrow$ $\nabla P_{0+} \neq (\mathbf{j} \times \mathbf{B})_{0+}$

3. **Loss of vertical position control of plasma magnetic field structure; Magnetic field rearranges, also** $\mathbf{u}_{0+} \neq 0$

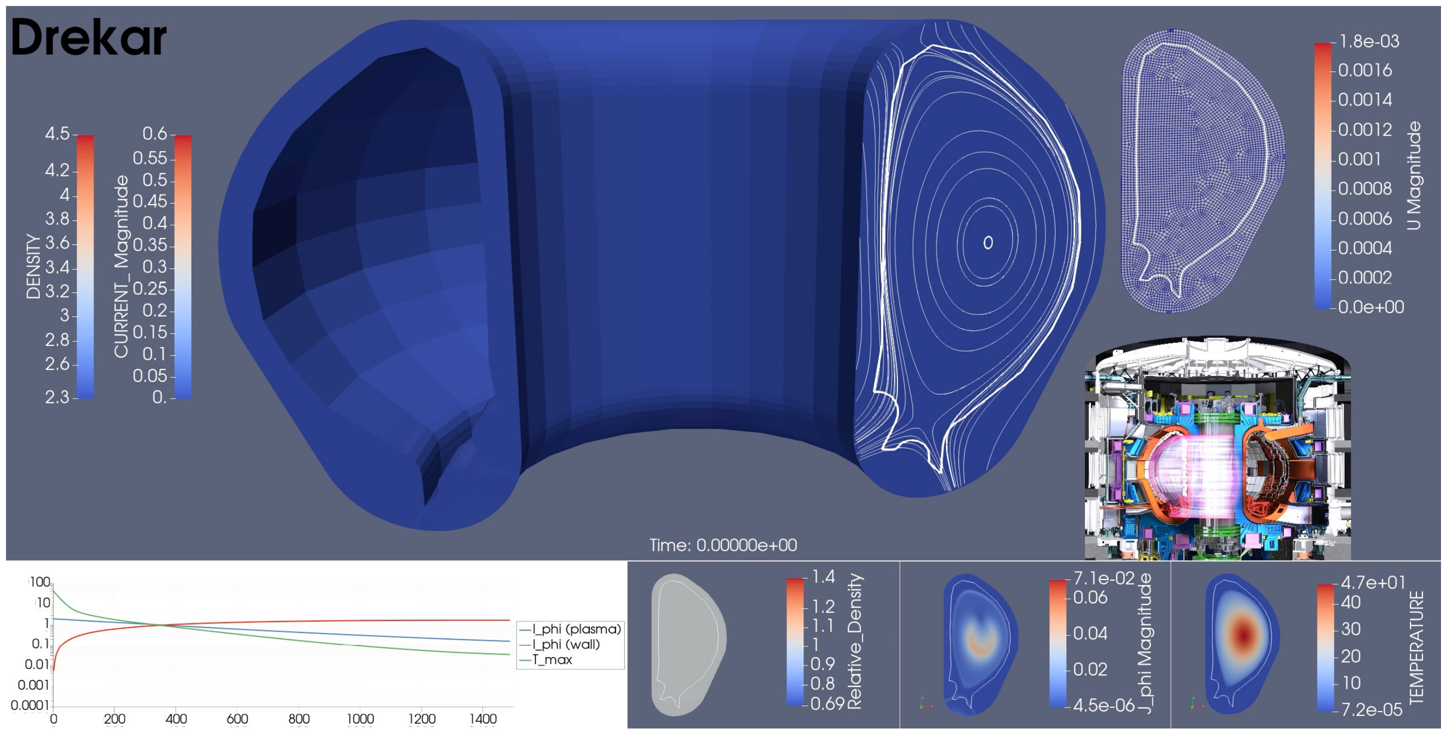
Computational Goals of Tokamak Disruption Simulation (TDS*) Center SciDAC-4 Partnership (DOE OFES/ASCR)

Cold VDE studies: Rapid loss of internal energy. Vertical displacement event (VDE) disruption simulation in ITER plasma and wall region. For our computations our initial choice was (dimensional parameters):

- ❑ **D_2 plasma**
- ❑ **T = 40 keV (~500M K)**
- ❑ **n = 10^{20} 1/m^3**
- ❑ **B_{max} = 8.5 T**
- ❑ **S = 10^4 (have run higher in VDE scaling > 10^{10})**

Proof-of-principle

Vertical displacement event (VDE) disruption simulation in ITER plasma and wall region.



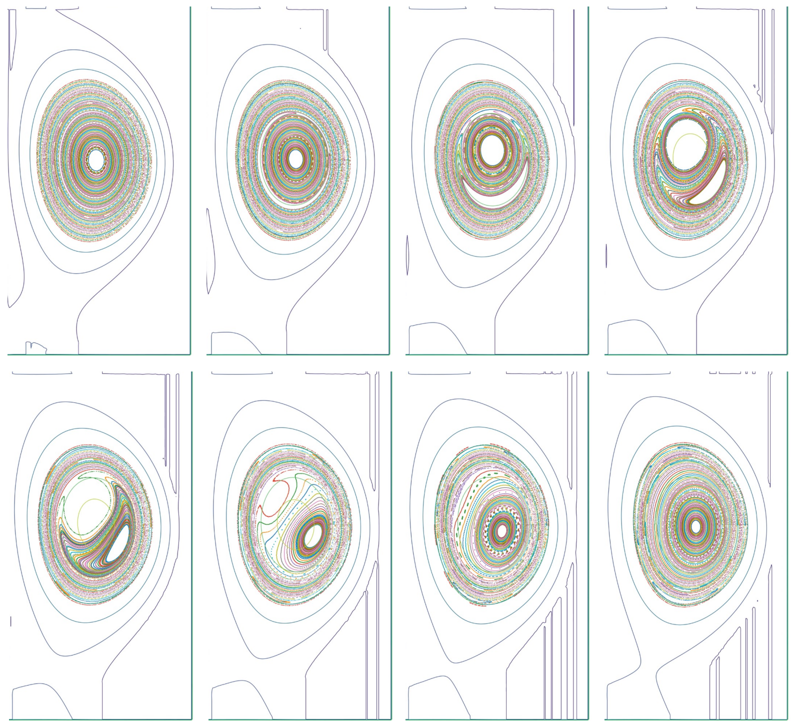
Bonilla, JS, Tang, Crockatt, Ohm, Phillips, Pawlowski, Conde, Beznosov, Comput. Methods Appl. Mechanics and Engrg. 2023

Internal kink

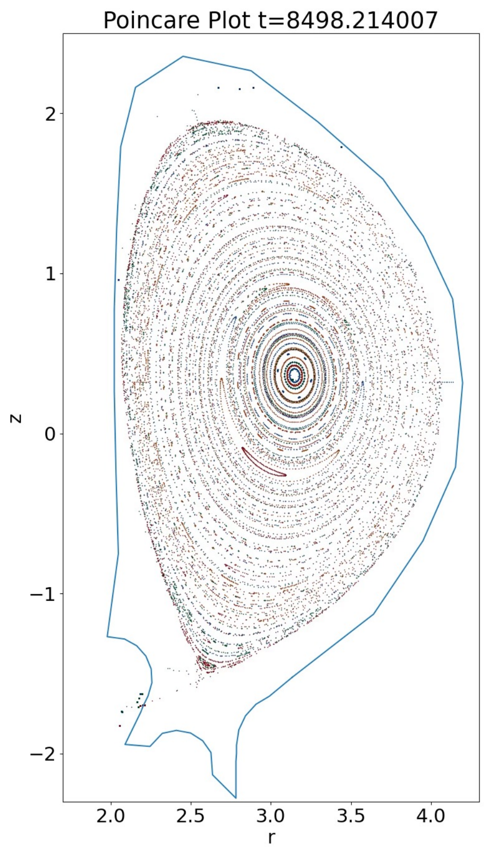
- Initial equilibrium with q profile < 0.9 (very unstable)
- Introduced (1,1) perturbation and let evolve in time
  - (1,1) leads to sawtooth crash with island growth
  - (2,1) is excited leading to stochastic magnetic field
  - breakdown of magnetic surfaces and a disruption.

Return map of B on poloidal plane: (Poincaré plot)

Sawtooth Progression



- B_0 : $5T$
- n : 10^{20}
- Sp : $10^4 - 10^6$
- $\kappa_\perp$: 10^{-5}
- $\kappa_\parallel$: 10^{-3}
- ν : 10^{-5}



Critical aspects of modeling disruptions and instabilities with MHD

- Need to integrate to longer time-scales a multiple time-scale multiphysics system
  - Convective transport, wave propagation (strong off-diagonal coupling), source terms, B involution, nearly incompressible flow (perpendicular to strong guide field)
- Whole device modeling (WDM) for fusion requires heterogeneous physics (plasma, wall, vacuum vessel).
- Higher-order temperature approximation is required for strongly anisotropic heat conductivity, and produces a mixed FE integration from i.e.

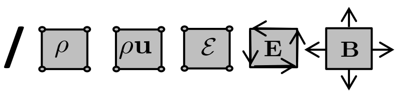
  First order cG FE for ($\mathbf{m}_h, \rho_h, \mathbf{B}_h, \psi_h$) and **second order** FE T_h.

  Note we have also demonstrated Q^2 for m_h, B_h as well. Q^N higher-order available.

- Important diffusion process for highly-resolved meshes (elliptic behavior)

Given these challenges we develop multiphysics block preconditioning approaches with AMG sub-block solvers to pursue development of optimal scalable solution methods

Approximate Block Factorization / Physics-based Preconditioning (using Teko)

- Applies to mixed interpolation (FE), staggered (FV), physics compatible / 

  structure preserving discretization using segregated unknown blocking

- Applies to heterogeneous physics systems (different physics in different sub-domains)

- Applies to systems where coupled system AMG is difficult or might completely fail
  (e.g. Hyperbolic systems with strong off diagonal physics coupling, multiphysics)

- Enables optimal AMG to be applied to sub-blocks (ML, MueLu)

- Handles disparate spatial discretizations and allows application of specialized
optimized AMG e.g. H(grad), H(curl), H(div) in the required spaces.

Structure of 5 x 5 block Jacobian system from Newton's method

$$\mathbf{F}'(\mathbf{x}_k)\mathbf{p} = -\mathbf{F}(\mathbf{x}_k)$$

$$\begin{bmatrix} \mathrm{F}_{\mathbf{m}} & \tilde{\mathrm{B}}^{\mathrm{T}}_{\mathbf{m},\boldsymbol{\rho}} & \mathbf{C}_{\mathbf{m},\mathbf{T}} & \mathbf{Z} & \mathbf{0} \\ \mathrm{B}_{\boldsymbol{\rho},\mathbf{m}} & \mathrm{F}_{\rho} & \mathbf{C}_{\rho,\mathbf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{\mathbf{T},\mathbf{m}} & \mathbf{C}_{\mathbf{T},\rho} & \mathrm{F}_{\mathrm{T}} & \mathbf{Z}_{\mathbf{T}} & \mathbf{0} \\ \mathbf{Y} & \mathbf{C}_{\mathbf{B},\rho} & \mathbf{0} & \mathbf{F}_{\mathbf{B}} & \mathbf{B}_{\mathbf{B}}{}^{T} \\ \mathbf{C}_{\boldsymbol{\psi},\mathbf{m}} & \mathbf{C}_{\boldsymbol{\psi},\rho} & \mathbf{0} & \mathbf{B}_{\mathbf{B}} & \mathbf{L}_{\boldsymbol{\psi}} \end{bmatrix} \begin{bmatrix} \delta\hat{\mathbf{m}} \\ \delta\hat{\rho} \\ \delta T \\ \delta\hat{\mathbf{B}} \\ \delta\hat{\boldsymbol{\psi}} \end{bmatrix} = - \begin{bmatrix} \mathbf{r}_{\mathbf{m}} \\ \mathrm{r}_{\rho} \\ \mathrm{r}_{T} \\ \mathbf{r}_{B} \\ \mathrm{r}_{\boldsymbol{\psi}} \end{bmatrix}$$

VMS FE resistive MHD and block solvers for MCF:

Ohm, Bonilla, Phillips, JS, Crockatt, Tuminaro, Hu, Tang, SISC, 2024

Bonilla, JS, Tang, Crockatt, Ohm, Phillips, Pawlowski, Conde, Beznosov, Comput. Methods Appl. Mech. Engrg. 2023

Teko multiphysics block preconditioning package:

E. C. Cyr, JS , and R. S. Tuminaro, SIAM SISC Vol. 38, No. 5, pp. S307–S331, 2016

$$[\mathbf{F_{ns}} = [\mathbf{F}_{\rho\mathbf{u}}, \mathrm{F}_\rho], \mathbf{F_B}, \mathrm{L}_r, \mathrm{F}_T]$$

Block Jacobi

$$\begin{bmatrix} \mathbf{F}_{\mathsf{ns}} & \mathbf{Z} & & C_T \\ \mathbf{Y} & \mathbf{F}_{\mathbf{B}} & \mathcal{B}_{\mathbf{B}}{}^T & \\ \mathbf{C}_{\mathsf{ns}} & \mathcal{B}_{\mathbf{B}} & \mathbf{L}_{\mathbf{r}} & \\ A_T & \mathbf{Z}_T & & \mathsf{F}_{\mathsf{T}} \end{bmatrix}^{-1}$$

$$\Big\downarrow$$

$$\begin{bmatrix} \begin{bmatrix} \mathbf{F}_{\mathsf{ns}} & \mathbf{Z} & \\ \mathbf{Y} & \mathbf{F}_{\mathbf{B}} & \mathcal{B}_{\mathbf{B}}{}^T \\ \mathbf{C}_{\mathsf{ns}} & \mathcal{B}_{\mathbf{B}} & \mathbf{L}_{\mathbf{r}} \end{bmatrix}^{-1} & \\ & [\mathsf{F}_{\mathsf{T}}]^{-1} \end{bmatrix}$$

$[\mathbf{F_{ns}} = [\mathbf{F}_{\rho\mathbf{u}}, \mathrm{F}_\rho], \mathbf{F_B}, \mathrm{L}_r, \mathrm{F}_T]$ Block Jacobi	
$[\mathbf{F}_{\mathsf{ns}}, \mathbf{F}_{\mathbf{B}}, \mathbf{L}_{\mathbf{r}}]$ Operator Splitting	$[\mathsf{F}_\mathsf{T}]$ AMG

$$
\begin{bmatrix}
\begin{bmatrix}
\mathbf{F}_{\mathsf{ns}} & \mathbf{Z} & \\
\mathbf{Y} & \mathbf{F}_{\mathbf{B}} & \mathcal{B}_{\mathbf{B}}{}^{T} \\
\mathbf{C}_{\mathsf{ns}} & \mathcal{B}_{\mathbf{B}} & \mathbf{L}_{\mathbf{r}}
\end{bmatrix}^{-1} & \\
 & [\mathsf{F}_\mathsf{T}]^{-1}
\end{bmatrix}
$$

<table>
<tr><td colspan="3">$[\mathbf{F}_{\mathbf{ns}} = [\mathbf{F}_{\rho\mathbf{u}}, \mathrm{F}_{\rho}], \mathbf{F}_{\mathbf{B}}, \mathrm{L}_{r}, \mathrm{F}_{T}]$
Block Jacobi</td></tr>
<tr><td colspan="2">$[\mathbf{F}_{\mathrm{ns}}, \mathbf{F}_{\mathbf{B}}, \mathbf{L}_{\mathbf{r}}]$
Operator Splitting</td><td>$[\mathrm{F}_{\mathrm{T}}]$
AMG</td></tr>
<tr><td>$[\mathbf{F}_{\mathbf{B}}, \mathbf{L}_{\mathbf{r}}]$
LU Decomp. + SIMPLEC</td><td>$[\mathbf{F}_{\mathbf{B}}, \mathbf{F}_{\mathrm{ns}}]$
LU Decomp. + SIMPLEC</td><td></td></tr>
</table>

$$\left(\begin{bmatrix} \mathbf{F}_{\mathrm{ns}} & \mathbf{Z} & \\ \mathbf{Y} & \mathbf{F}_{\mathbf{B}} & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{F}_{\mathbf{B}}^{-1} & \\ \mathbf{C}_{\mathrm{ns}} & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{F}_{\mathbf{B}} & \mathcal{B}_{\mathbf{B}}^{T} \\ & \mathcal{B}_{\mathbf{B}} & \mathbf{L}_{\mathbf{r}} \end{bmatrix} \right)^{-1}$$

2x2 critical implicit Stiff Alfven wave coupling

2x2 Saddle point system for (B,r)

Ohm, Bonilla, Phillips, JS, Crockatt, Tuminaro, Hu, Tang, SIAM SISC, 2024

$[\mathbf{F_{ns}} = [\mathbf{F}_{\rho\mathbf{u}}, \mathrm{F}_\rho], \mathbf{F_B}, \mathrm{L}_r, \mathrm{F}_T]$
Block Jacobi

$[\mathbf{F}_{\mathrm{ns}}, \mathbf{F_B}, \mathbf{L_r}]$ Operator Splitting	$[\mathrm{F_T}]$ AMG

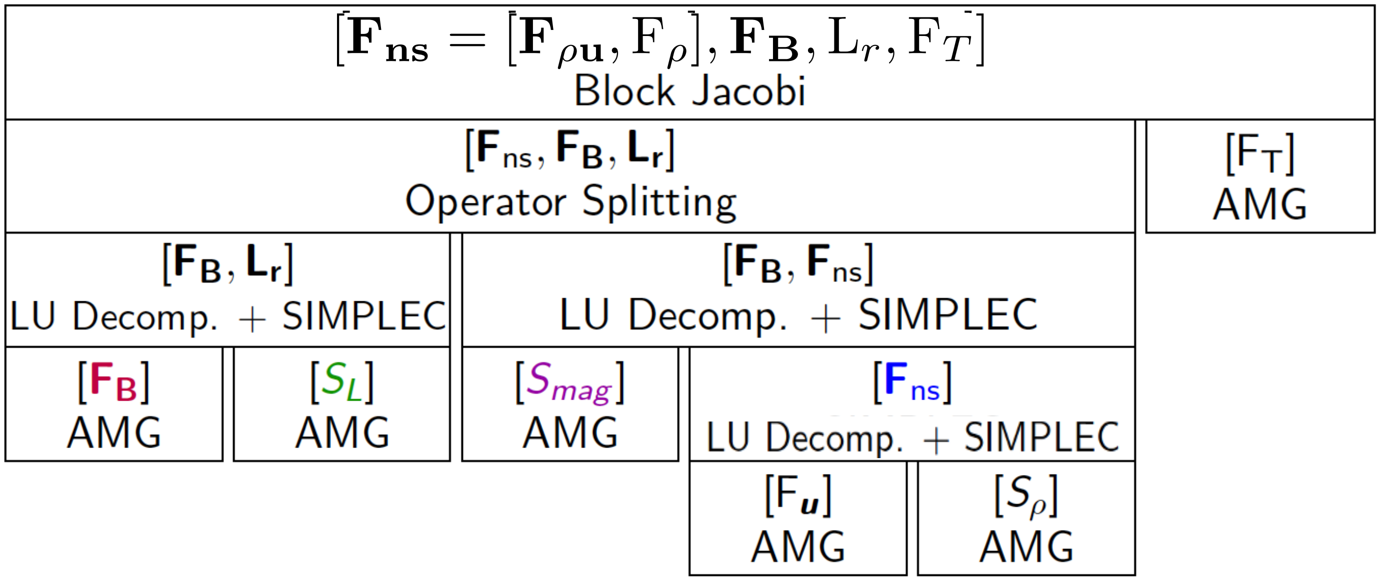
$[\mathbf{F_B}, \mathbf{L_r}]$ LU Decomp. + SIMPLEC	$[\mathbf{F_B}, \mathbf{F}_{\mathrm{ns}}]$ LU Decomp. + SIMPLEC

$[\mathbf{F_B}]$ AMG	$[S_L]$ AMG	$[S_{mag}]$ AMG	$[\mathbf{F}_{\mathrm{ns}}]$ LU Decomp. + SIMPLEC

$$\begin{bmatrix} \mathbf{I} & & \\ & \mathbf{F_B} & \mathcal{B}_{\mathbf{B}}{}^T \\ & & S_L \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{F_B} & \\ & -\mathcal{B}_{\mathbf{B}} & \mathbf{I} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{F}_{\mathrm{ns}} & \mathbf{Z} & \\ & S_{mag} & \\ & & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & & \\ -\mathbf{Y}\,\mathbf{F}_{\mathrm{ns}}{}^{-1} & \mathbf{I} & \\ & & \mathbf{I} \end{bmatrix}$$

$$S_L := \mathbf{L_r} - \mathcal{B}_{\mathbf{B}}(\mathrm{absrowsum}(\mathbf{F_B}))^{-1}\,\mathcal{B}_{\mathbf{B}}{}^T, \qquad S_{mag} := \mathbf{F_B} - \mathbf{Y}(\mathrm{absrowsum}(\mathbf{F}_{\mathrm{ns}}))^{-1}\,\mathbf{Z}$$

$$\mathbf{F}_{ns} = \begin{bmatrix} \mathbf{F}_{\rho u} & \mathbf{B}^T_{\rho u,\rho} \\ \mathbf{B}_{\rho,\rho u} & \mathbf{F}_{\rho} \end{bmatrix}$$



$$\mathbf{F}_{\mathrm{ns}} \approx \mathcal{M}_{NS} = \begin{bmatrix} \mathrm{F}_{\boldsymbol{\rho u}} & \tilde{\mathrm{B}}^{\mathrm{T}}_{\boldsymbol{\rho u},\rho} \\ & S_{\rho} \end{bmatrix}$$

$$S_{\rho} = \mathrm{F}_{\rho} - \mathrm{B}_{\rho,\boldsymbol{\rho u}}(\mathrm{absrowsum}(\mathrm{F}_{\boldsymbol{\rho u}}))^{-1}\, \tilde{\mathrm{B}}^{\mathrm{T}}_{\boldsymbol{\rho u},\rho}$$

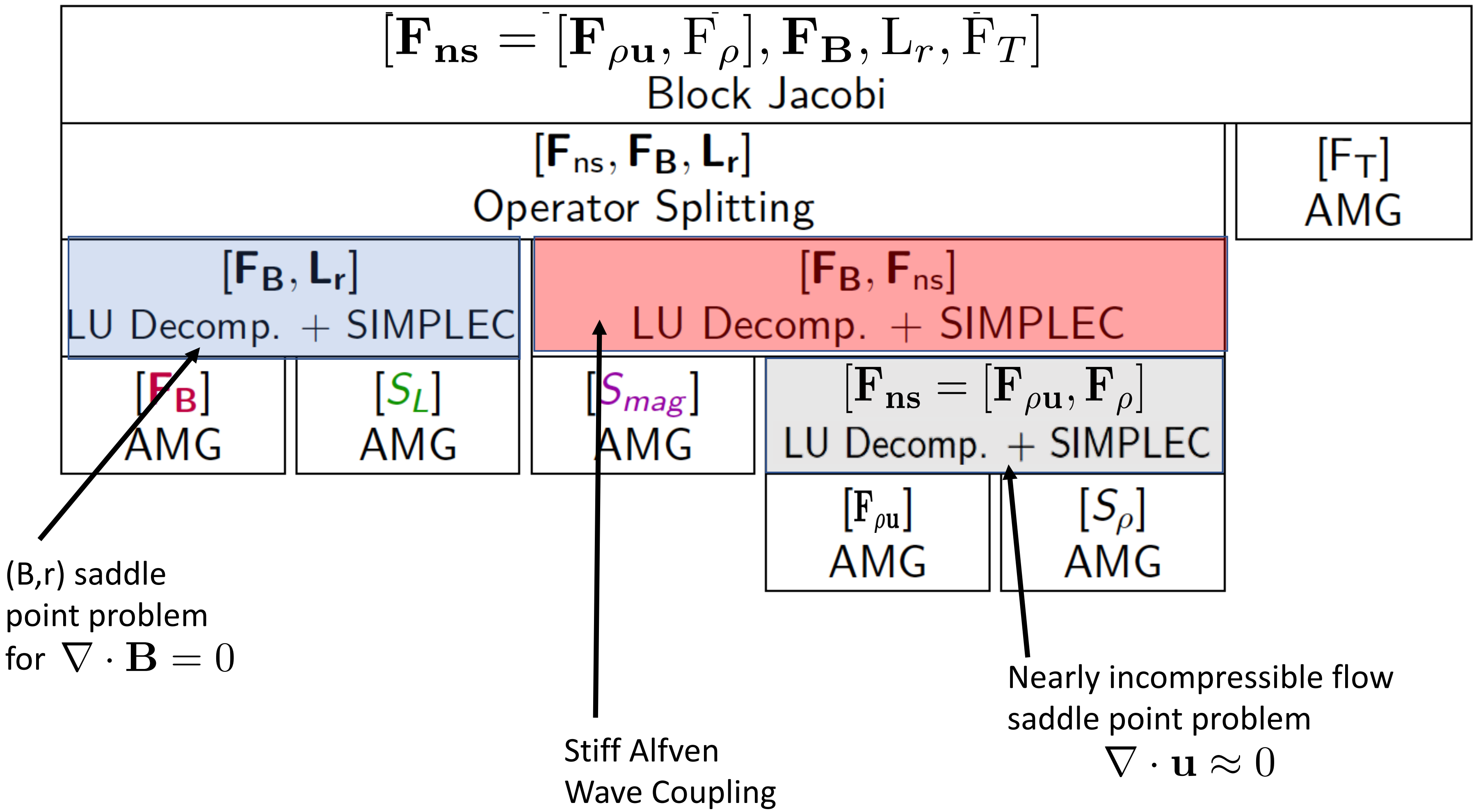
Kay, Loghin, Wathen, Silvester, Elman (1999 - 2006); (PCD etc.)

Pernice and Tocci (2001);

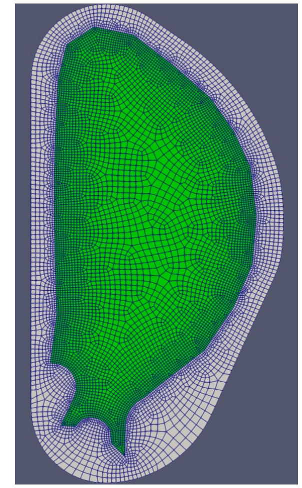
Elman, Howle, JS, Tuminaro (2003);

(Taxonomy of Block Preconditioners, Elman, Howle, JS, Shuttleworth, Tuminaro, JCP 2008);

Ohm, Bonilla, Phillips, JS, Crockatt, Tuminaro, Hu, Tang, SIAM SISC, 2024


[Fns = [Fρu, Fρ], FB, Lr, FT]
Block Jacobi
[Fns, FB, Lr]
Operator Splitting
[FT]
AMG
[FB, Lr]
LU Decomp. + SIMPLEC
[FB, Fns]
LU Decomp. + SIMPLEC
[FB]
AMG
[SL]
AMG
[Smag]
AMG
[Fns = [Fρu, Fρ]
LU Decomp. + SIMPLEC
[Fρu]
AMG
[Sρ]
AMG
(B,r) saddle point problem for ∇ · B = 0
Stiff Alfven Wave Coupling
Nearly incompressible flow saddle point problem ∇ · u ≈ 0

Drekar Strong Scaling Results 3D ITER VDE

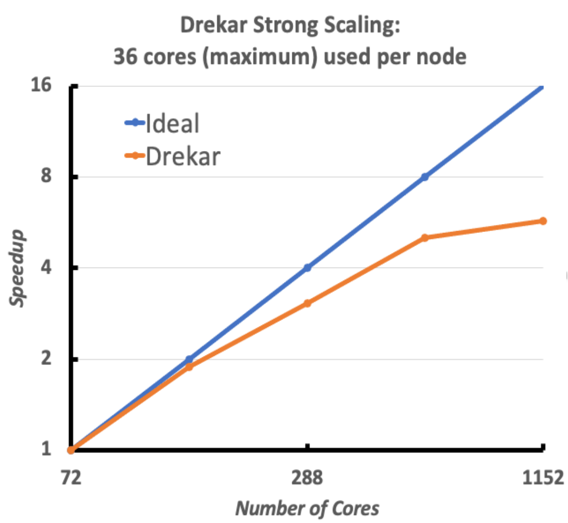


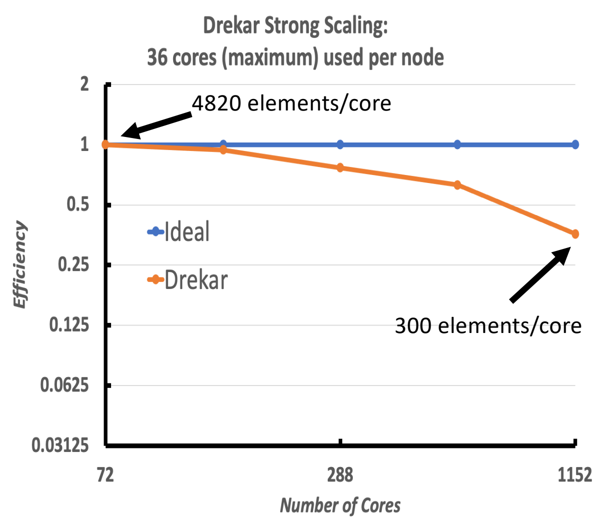
Strong scaling unperturbed initial equilibrium on coarse mesh with S = 1e4.

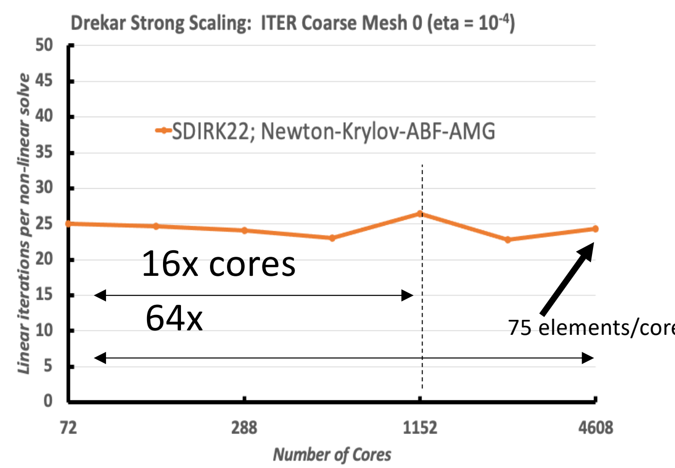
To 250 global Alfven times. 72 cores → 4608 cores (64x increase).

Coarse Mesh 0: 347K elements, 7.2K poloidal x 48 toroidal

Constant time-step size dt = 2
Max CFL_A ~ 750, CFL_u ~3.2

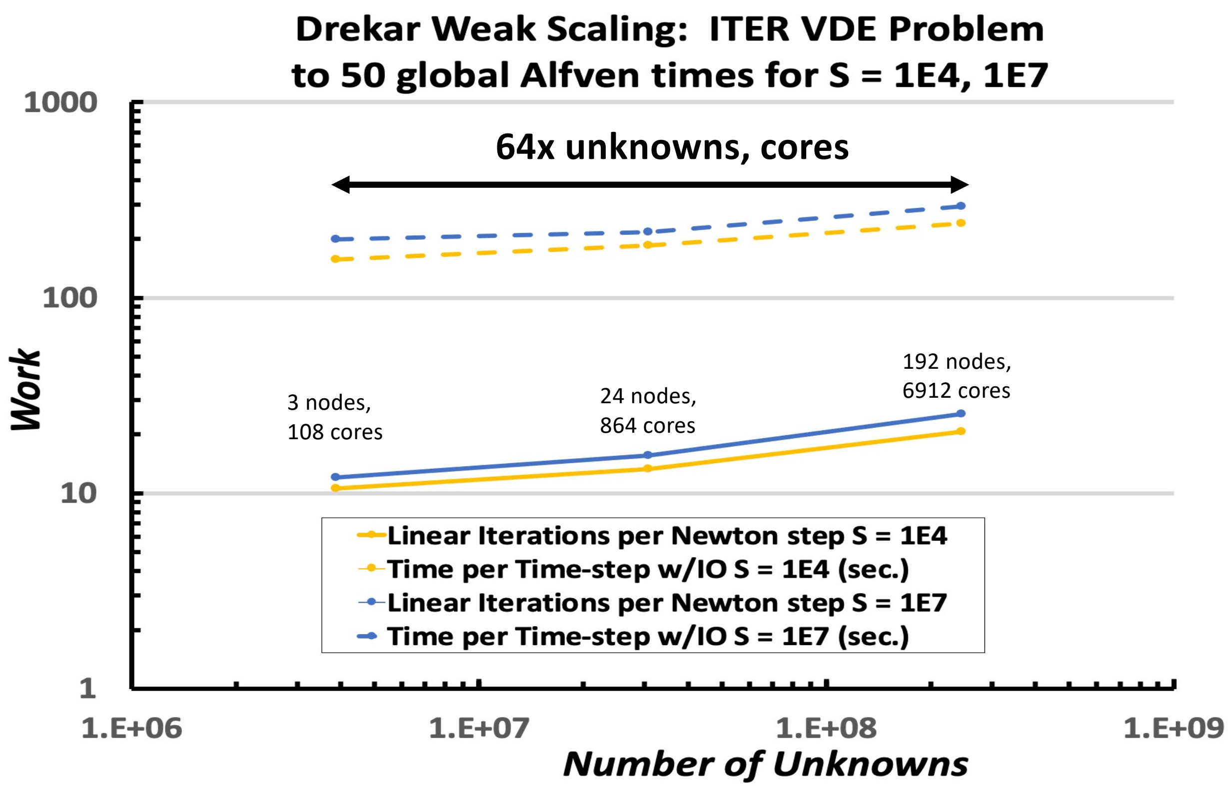




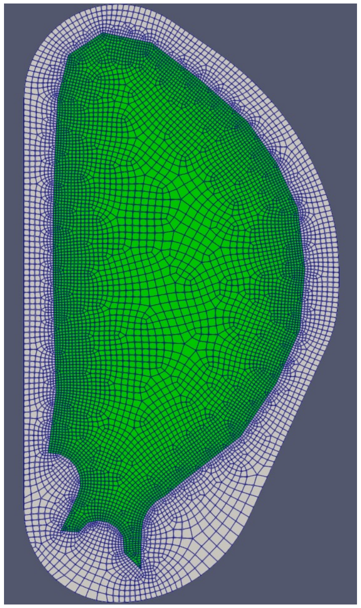


Outer iterations GMRES, Blocks –
AMG one V-cycle; smoothers DD/ILU(0)

Drekar Weak scaling Study 3D ITER VDE




```
mesh 0. : 347K  elements,  7.2K poloidal x  48 toroidal, dt ~ 1.2;  108  cores on ghost
mesh 1. : 2.77M elements, 28.8K poloidal x  96 toroidal, dt ~ 0.55; 864  cores on ghost
mesh 2. : 22.2M elements,115.2K poloidal x 192 toroidal, dt ~ 0.25; 6912 cores on ghost
```



Unstructured
mesh sequence

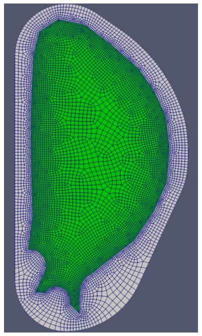
To 50 global Alfven times

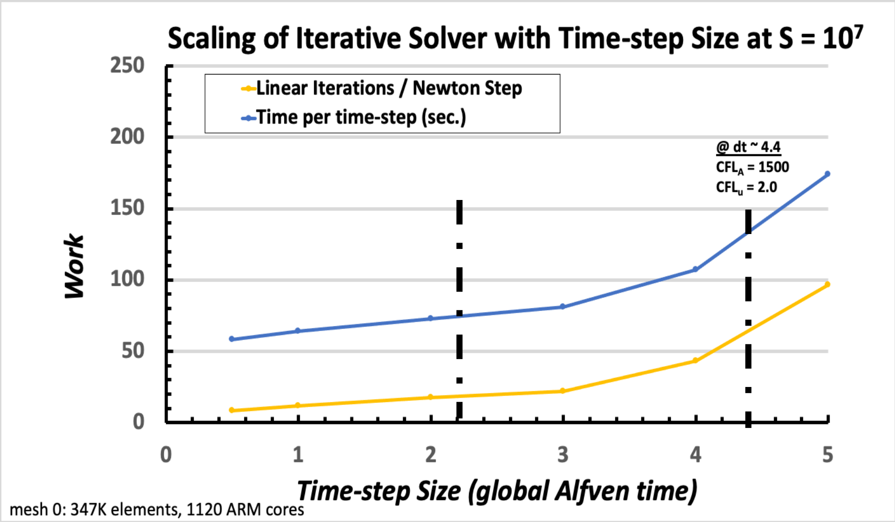
Max CFL_A = 400, CFL_u = 2

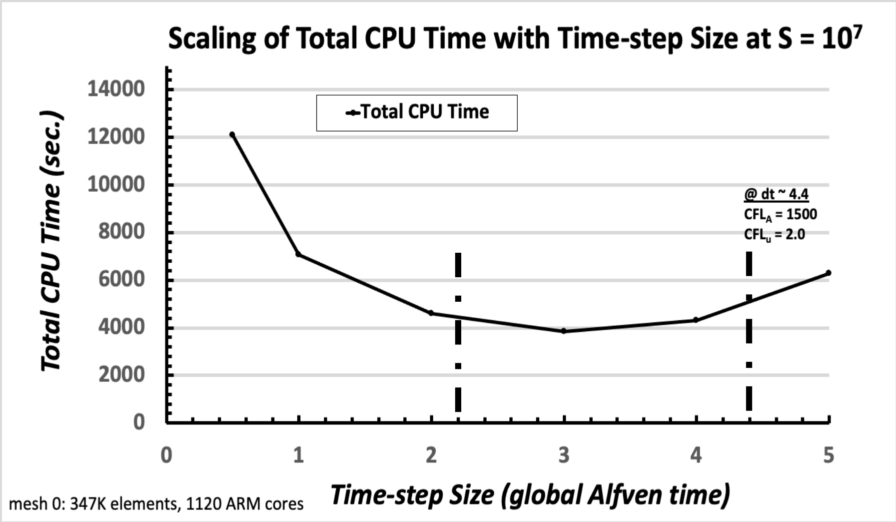
Pr_m = 0.1, Pr_T = 1

Time-step Size scaling for S = 10^7

Coarse mesh0



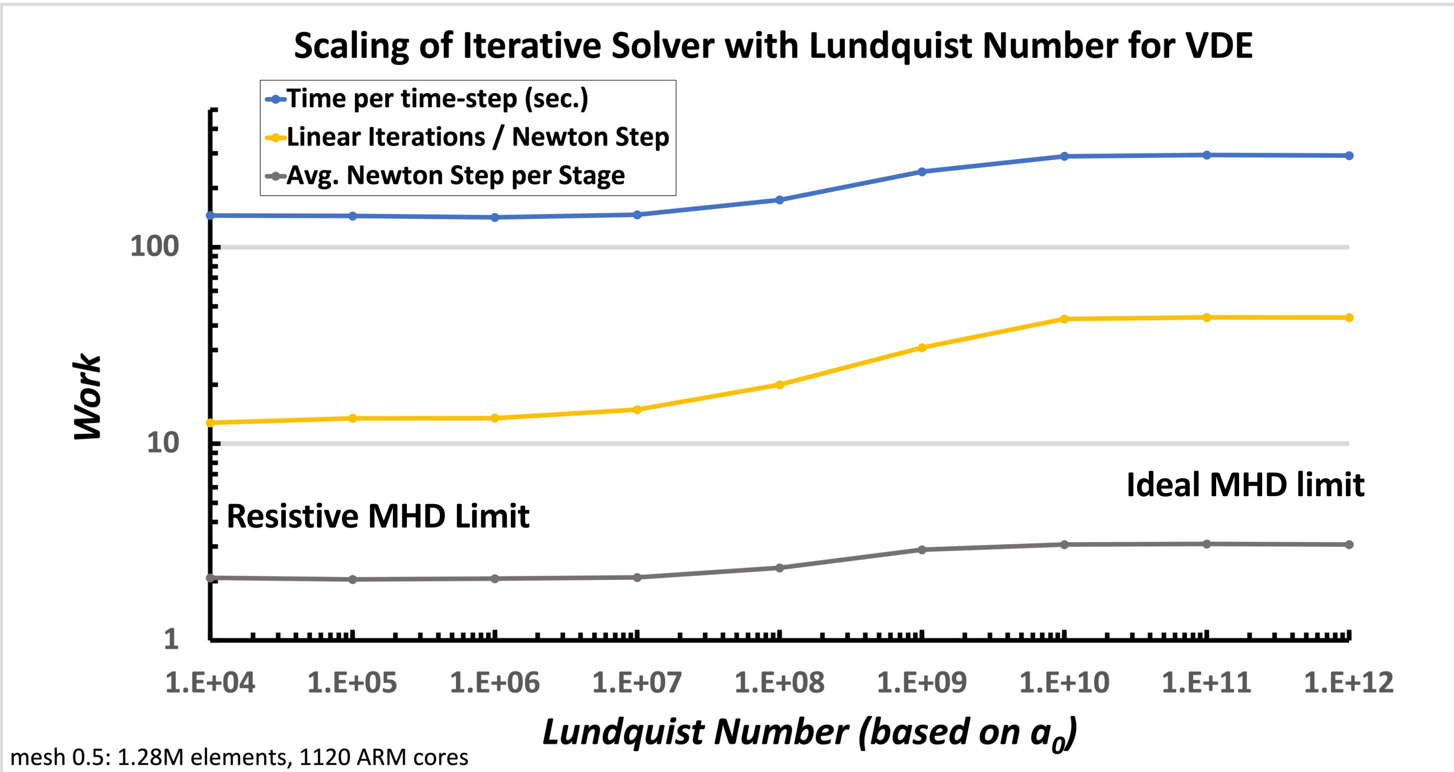


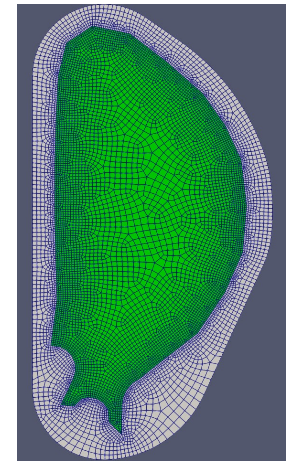


To 100 global Alfven times

Lundquist Number scaling, coarse mesh 0.5 to 25 global Alfven times.

(Pr_m = 10, Pr_T = 1 i.e. both momentum and thermal diffusivities the same, scale 10x resistivity)





```
Mesh 0.5:
1.28M elements,10K
poloidal x 128
toroidal, 1120 ARM
cores
```

To 25 global Alfven times

Max CFL_A = 400, CFL_u = 2

dt ~ 1.0

Conclusions

- Developed scalable fully implicit low Mach compressible visco-resistive MHD solver.
  - VMS FE → pursuing the control of numerical instabilities (convection, unresolved gradients) and saddle point system solvability (demonstrated numerically)
  - Approximate block factorization → scalable treatment of multiphysics equation coupling.
  - Scalable AMG solves → efficient scalable nearly optimal sub-block preconditioning.
- Demonstrated scalability (strong and weak), Lundquist number robustness, promising initial efficiency for longer-time scale simulations.
- Proof-of-principle numerical experiments.
  - Cold VDE and a (1,1) internal kink mode.
- Future work
  - Implement more complete reuse of AMG projections and all symbolic factorizations
  - Complete weak implementation of tangential interface conditions on B
  - Further V&V benchmarks.
  - Extension to two temperature models (T_i, T_e).
  - Extended MHD (XMHD) formulation.